Journal of Statistical Physics, Vol. 46, Nos. 3/4, 1987

Existence and Uniqueness of Gibbs States for a Statistical Mechanical Polyacetylene Model

Yong Moon Park[1]

Received May 20, 1986

One-dimensional polyacetylene is studied as a model of statistical mechanics. In a semiclassical approximation the system is equivalent to a quantum XY model interacting with unbounded classical spins in one-dimensional lattice space Z. By establishing uniform estimates, an infinite-volume-limit Hilbert space, a strongly continuous time evolution group of unitary operators, and an invariant vector are constructed. Moreover, it is proven that any infinite-limit state satisfies Gibbs conditions. Finally, a modification of Araki's relative entropy method is used to establish the uniqueness of Gibbs states.

KEY WORDS: Polyacetylene; Gibbs states; KMS states; Choquet simplex; Araki's relative entropy method; Dyson expansion; ground states.

1. INTRODUCTION

We study the (quantum) statistical mechanics of a polyacetylene model. Polyacetylene $(CH)_x$ is a one-dimensional polymer, which exhibits some interesting properties. The following Hamiltonian for a bounded region $\Lambda = \{-n, -n+1, \dots, n\} \subset Z$ was suggested by Schrieffer *et al.*[10]:

$$H = H_B + H_F \tag{1.1}$$

with

$$H_B = \sum_{i=-n}^{n} \frac{1}{2M} P_i^2 + \sum_{i=-n}^{n-1} \frac{w^2}{2} (u_i - u_{i+1})^2$$

$$H_F = \sum_{i=-n}^{n-1} \sum_{s=\pm 1/2} [C_{s,i}^* C_{s,i+1} + C_{s,i+1}^* C_{s,i}][t - g(u_i - u_{i+1})]$$

[1] Department of Mathematics, Yonsei University, Seoul 120, Korea.

0022-4715/87/0200-0567$05.00/0

where M is the mass of CH, w is a positive constant, and t and g are real constants. The indices $s = \pm\frac{1}{2}$ stand for the spins of the fermions in C–C bonds, and one imposes

$$\begin{aligned} [P_i, u_j] &= -i\delta_{ij} \\ \{C^*_{s,i}, C_{s',j'}\} &= \delta_{ss'}\delta_{ij'} \end{aligned} \tag{1.2}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. For detailed discussions of the model, we refer to Refs. 4, 5, 10, and 11.

Some heuristic arguments suggest that the vibrations $(-1)^x(u_x - u_{x+1})$ take the form of solitons, and that fractional charges may appear under the influence of the solitions.[10,11] In Ref. 5 we tried to clarify these heuristic arguments with rigorous statistical mechanics, and we established some properties, such as the exponential clustering of some fermion correlation functions. Our purpose in this paper is to construct the thermodynamic limit theory and to establish the uniqueness and the cluster property of infinite-volume Gibbs states of the model. Thus, our results are extensions and complements of those in Ref. 5. The results show that the previous heuristic arguments in Refs. 10 and 11 need serious reconsideration at least for $\beta < \infty$ (and under a semiclassical approximation).

Instead of the full model (1.1), we make a semiclalsical approximation to simplify our discussions:

Assumption A. (A semiclassical approximation) M in (1.1) is so large so that the boson kinetic energy term is negligible.

Under the above assumption, we introduce new boson fields

$$\phi_{i+1/2} = -(u_i - u_{i+1})$$

whose correlation functions are well-defined. We regard $\{\phi_{i+1/2}\}$ as independent boson fields. That is, we impose free boundary conditions for the boson fields and then make the changes of variables $-(u_i - u_{i+1}) \to \phi_{i+1/2}$. Since there is no direct coupling between $s = 1/2$ and $s = -1/2$ fermions, we consider one kind of fermion. Therefore, for any $\Lambda = \{n, n+1,..., m\} \subset Z$ we write

$$H_\Lambda = H_{\Lambda,\mathrm{B}} + H_{\Lambda,\mathrm{F}} \tag{1.3}$$

with

$$H_{\Lambda,\mathrm{B}} = \sum_{i=n}^{m} \frac{w^2}{2} \phi^2_{i+1/2} \tag{1.4}$$

$$H_{\Lambda,\mathrm{F}} = \sum_{i=n}^{m-1} [C^*_i C_{i+1} + C^*_{i+1} C_i](t + g\phi_{i+1/2}) \tag{1.5}$$

where

$$\{C_i^*, C_j\} = \delta_{ij}, \qquad \{C_i, C_j\} = 0 \tag{1.6}$$

That is, the fermions satisfy the anticommutation relations (CAR).

We next discuss the algebra of observables. Let us denote

$$\mathscr{A}_\Lambda^{\mathrm{F}} = \text{the } C^*\text{-algebra generated by } \{C_i, C_i^*;\quad i \in \Lambda\}$$

$$\mathscr{A}_\Lambda^{\mathrm{B}} = \text{the } C^*\text{-algebra of bounded continuous functions of variables} \tag{1.7}$$

$$\phi_{i+1/2}, \quad i \in \Lambda$$

$$\mathscr{A}_\Lambda = \mathscr{A}_\Lambda^{\mathrm{B}} \otimes \mathscr{A}_\Lambda^{\mathrm{F}} \tag{1.8}$$

The quasilocal algebra of observables is given by

$$\mathscr{A} = \mathscr{A}^{\mathrm{B}} \otimes \mathscr{A}^{\mathrm{F}} \tag{1.9}$$

with

$$\mathscr{A}^{\mathrm{B}} = \overline{\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda^{\mathrm{B}}}, \qquad \mathscr{A}^{\mathrm{F}} = \overline{\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda^{\mathrm{F}}}$$

The finite-volume Gibbs states and the partition function are defined by

$$\begin{aligned} \rho_\Lambda(A) &= \frac{1}{Z_\Lambda} \int \prod_{i=n}^{m} d\phi_{i+1/2}\, e^{-\beta H_{\Lambda,\mathrm{B}}}\, \mathrm{Tr}_{F_\Lambda}(A e^{-\beta H_{\Lambda,\mathrm{F}}}) \\ Z_\Lambda &= \int \prod_{i=n}^{m} d\phi_{i+1/2}\, e^{-\beta H_{\Lambda,\mathrm{B}}}\, \mathrm{Tr}(e^{-\beta H_{\Lambda,\mathrm{F}}}) \end{aligned} \tag{1.10}$$

where $\beta > 0$, $A \in \mathscr{A}_\Lambda$, $F_\Lambda = C^{|\Lambda|}$, and $H_{\Lambda,\mathrm{B}}$ and $H_{\Lambda,\mathrm{F}}$ are given by (1.4).

In order to construct infinite-volume-limit equilibrium states we introduce Green's functions.[2] Let α^Λ be the time evolution automorphism on $\mathscr{A}_\Lambda$ given by

$$\alpha_t^\Lambda(A) = e^{itH_{\Lambda,\mathrm{F}}} A e^{-itH_{\Lambda,\mathrm{F}}} \tag{1.11}$$

The finite-volume Green's functions are given by

$$G_\Lambda(A, B; t) = \rho_\Lambda(A \alpha_t^\Lambda(B)) \tag{1.12}$$

Although ρ_Λ is defined as a state over $\mathscr{A}_\Lambda$, it has an extension to a state on $\mathscr{A}$ by the Hahn–Banach theorem, which we denote again ρ_Λ. The bounds

$$|G_\Lambda(A, B; t)| \leqslant \|A\|\, \|B\| \tag{1.13}$$

imply that there exists a subnet $\{\Lambda_\alpha\}$ such that

$$G(A, B; t) = \lim_{\Lambda_\alpha \to Z} G(A, B; t) \tag{1.14}$$

for all $A, B \in \mathscr{A}$ and $t \in R$. This is a consequence of Tychonoff's theorem. Clearly, the value

$$\rho(A) = G(A, 1; 0) \tag{1.15}$$

determines a state ρ over the quasilocal algebra $\mathscr{A}$. We write that for any finite sequence $h: Z + 1/2 \to R$,

$$\phi(h) = \sum_{x \in Z} \phi_{x+1/2}\, h(x + 1/2) \tag{1.16}$$

For any state ω on $\mathscr{A}$, let $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic GNS representation of $\mathscr{A}$ with respect to ω. Let $\pi_\omega(\phi(h))$ be the generator of the group $\pi_\omega(\exp[it\phi(h)]$. We say that a state ω on $\mathscr{A}$ is an entire analytic state if $(\Omega_\omega, \exp[z\pi_\omega(\phi(h))]\Omega_\omega)$ is an entire analytic function for each finite sequence h.

We now list our main results in this paper.

Theorem 1.1. Let ρ be any weak*-limit state of finite-volume Gibbs states ρ_Λ defined as in (1.15). Then ρ is an entire analytic state on $\mathscr{A}$. Let $(\mathscr{H}_\rho, \pi_\rho, \Omega_\rho)$ be the cyclic representation with respect to ρ. Then there exists an essentially self-adjoint operator H defined on $\pi_\rho(\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda)$ such that

$$G(A, B; t) = (\pi_\rho(A^*)\Omega_\rho, e^{itH}\pi_\rho(B)\, e^{-itH}\Omega_\rho)$$

for any $A, B \in \mathscr{A}$, $t \in R$. Moreover, Ω_ρ is invariant under e^{itH}.

We will prove the above result in Section 2. We next define Gibbs conditions for the model. For any finite $\Lambda = \{n, n+1, ..., m\} \subset Z$, let

$$\begin{aligned} W_\Lambda = {} & [C^*_{n-1} C_n + C^*_n C_{n-1}](t + g\phi_{n-1/2}) \\ & + [C^*_m C_{m+1} + C^*_{m+1} C_m](t + g\phi_{m+1/2}) \end{aligned} \tag{1.17}$$

Thus, W_Λ is the surface energy of Λ. For a given entire analytic state ω on $\mathscr{A}$ we define $\pi_\omega(W_\Lambda)$ by replacing C_x and $\phi_{x+1/2}$ by $\pi_\omega(C_x)$ and $\pi_\omega(\phi_{x+1/2})$, respectively, in (1.17). For any finite sequence $h: Z + 1/2 \to R$, let

$$\|h\|_P = \left[\sum_{x \in Z} |h(x + 1/2)|^P\right]^{1/P} \tag{1.18}$$

We now list Gibbs conditions for a state ω on $\mathscr{A}$:

Definition 1.2. For given $\beta > 0$ and w, t, $g \in R$, let Γ be the set of states on $\mathscr{A}$ statisfying the following conditions:

(G-1) [Regularity] Let $c = \max\{1/2\beta w^2, 2\beta g/w^2\}$. Then each $\omega \in \Gamma$ is entire analytic and satisfies

$$\omega(\exp[\phi(h)]) \leqslant \exp\{c[\|h\|_2^2 + \|h\|_1]\}$$

uniformly in $\omega \in \Gamma$.

(G-2) Let $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic representation with respect to $\omega \in \Gamma$. Then for each $\omega \in \Gamma$, Ω_ω is a separating vector for $\pi_\omega(\mathscr{A})''$.

(G-3) Let h be the generator of the modular automorphism group for $\omega \in \Gamma$. Then for any $B \in \mathscr{A}_\Lambda$, $\Lambda \subset Z$,

$$h\pi_\omega(B)\Omega_\omega = [\pi_\omega(H_\Lambda + W_\Lambda), \pi_\omega(B)]\Omega_\omega$$

(G-4) [Gibbs conditions] For any finite $\Lambda \subset Z$, let $\mathscr{A}_{\Lambda^c}$ be the closure of $\bigcup_{\Lambda \cap \Lambda' = \varnothing} \mathscr{A}_{\Lambda'}$. Let

$$\Gamma^\Lambda_{i\beta/2}\Omega_\omega = \exp\{-\tfrac{1}{2}\beta[H - \pi_\omega(W_\Lambda)]\}\Omega_\omega$$

Then for $\omega \in \Gamma$

$$(\Gamma^\Lambda_{i\beta/2}\Omega_\omega, \pi_\omega(A)\,\pi_\omega(B)\,\Gamma^\Lambda_{i\beta/2}\Omega_\omega) = \rho_\Lambda(A)(\Gamma^\Lambda_{i\beta/2}\Omega_\omega, \pi_\omega(B)\,\Gamma^\Lambda_{i\beta/2}\Omega_\omega)$$

for any $A \in \mathscr{A}_\Lambda$, $B \in \mathscr{A}_{\Lambda^c}$, where ρ_Λ is the finite-volume Gibbs state. We say that any $\omega \in \Gamma$ is a *Gibbs state.*

Remark. (a) The regularity condition (G-1) says that each $\omega \in \Gamma$ is an entire analytic state and so (G-3) makes sense.

(b) The expression in (G-4) is a formal expression. In Section 3 (and Section 4) we will give a precise meaning to $\exp\{-\frac{1}{2}\beta[H - \pi_\omega(W_\Lambda)]\}\Omega_\omega$ for any Gibbs state $\omega \in \Gamma$ via a Dyson expansion.[1,2] See Propositions 3.5 and 4.1.

We then have the following result:

Theorem 1.3. Let ρ be any weak*-limit of $\{\rho_\Lambda\}$ and let H be the self-adjoint operator in Theorem 1.1. Then $H = h$, and ρ is a Gibbs state.

We will prove Theorem 1.3 in Section 3.

In order to discuss uniqueness of Gibbs states and the clustering property, we introduce a C^*-subalgebra $\mathscr{A}^e$ of $\mathscr{A}$ as follows. Let

$$\mathscr{A}_\Lambda^{F,e} = \text{the algebra generated by even monomials in } C_i^* \text{ and } C_j, \quad i, j \in \Lambda \tag{1.19}$$

$$\mathscr{A}_\Lambda^e = \mathscr{A}_\Lambda^{\mathrm{B}} \otimes \mathscr{A}_\Lambda^{\mathrm{F},e}$$

and

$$\mathscr{A}^e = \overline{\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda^e} \tag{1.20}$$

For each $j \in Z$, let $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$ be 2×2 Pauli matrices,[2] and let $\hat{\mathscr{A}}_\Lambda^P$ be the algebra generated by $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$, $j \in \Lambda$. Similarly, let $\hat{\mathscr{A}}_\Lambda^{P,e}$ be the algebra generated by even monomials in $\sigma_j^{(1)}$ and $\sigma_k^{(2)}$, $j, k \in \Lambda$. We write $\hat{\mathscr{A}}$ and $\hat{\mathscr{A}}^e$ for the closure of $\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda^{\mathrm{B}} \otimes \hat{\mathscr{A}}_\Lambda^P$ and $\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda^{\mathrm{B}} \otimes \hat{\mathscr{A}}_\Lambda^{P,e}$, respectively. It is known that

$$\mathscr{A}^e = \hat{\mathscr{A}}^e \tag{1.21}$$

(see Example 6.2.14 of Ref. 2 and references therein), and that[2,5]

$$H_{\Lambda,\mathrm{F}} = -2 \sum_{j=n}^{m-1} (\sigma_j^{(1)}\sigma_{j+1}^{(1)} + \sigma_j^{(2)}\sigma_j^{(2)})(t + g\phi_{j+1/2}) \tag{1.22}$$

Thus, the model is equivalent to a quantum XY model interacting with boson fields.

Each element $A \in \mathscr{A}$ has a unique decomposition into even and odd parts

$$A = A^+ + A^-$$

where $A^+ \in \mathscr{A}^e$.[2] Let $\mathscr{A}^0$ be the space of odd elements. We say that a state ω is an *even Gibbs state* if it is a Gibbs state and if $\omega(A) = \omega(A^+)$ for any $A \in \mathscr{A}$.

Theorem 1.4. The set Γ^e of even Gibbs states consists of one element.

Remark. (a) By the definition of finite-volume Gibbs states ρ_Λ in (1.10), it turns out that any weak*-limit ρ of finite Gibbs states ρ_Λ is an even Gibbs state.

(b) Using the Gibbs condition (G-4), one may show that any Gibbs state is even. See the remark below the proof of Lemma 4.2 in Section 4.

In Section 4, we will show that the set Γ^e of even Gibs states on $\mathscr{A}^e$ forms a metrizable (convex and compact) simplex, and that for any extremal state $\omega \in \Gamma^e$, the algebra $\mathscr{B}^e_\omega$ at infinity for $(\mathscr{A}^e, \omega)$ is trivial. As a consequence of Theorem 1.4, we have the following (see, e.g., Theorem 2.6.10 of Ref. 2):

Theorem 1.5. The unique Gibbs state ω has the cluster property: For any $A \in \mathscr{A}^e_\Lambda$, $B \in \mathscr{A}^e_{\Lambda'}$,

$$|\omega(AB) - \omega(A)\,\omega(B)| \to 0$$

as $\operatorname{dist}(\Lambda, \Lambda') \to \infty$.

It may be worthwhile to comment on Theorem 1.5 (and also on Theorem 1.4). For a technical reason (Proposition 4.4), we are unable to extend Theorem 1.5 to $\mathscr{A}$. The difficulty comes from complicated local structures of $\mathscr{A}$, i.e., $[\mathscr{A}_{\Lambda'}, \mathscr{A}_\Lambda] \neq 0$ even if $\Lambda \cap \Lambda' = \varnothing$. If one can show the result in Proposition 4.4 for $T \in \mathscr{Z}_\omega$, $A \in \pi_\omega(\mathscr{A}_\Lambda)$, and $B \in \pi_\omega(\mathscr{A}_{\Lambda^c})$, where $\mathscr{Z}_\omega$ is the center of $\pi_\omega(\mathscr{A})''$, then the restrictions to even elements in our results can be removed.

The contents of this paper are as follows: In Section 2, we establish some uniform estimates (Propositions 2.1–2.3) for finite Gibbs states. Using the uniform estimates, we prove Theorem 1.1. In Section 3, we introduce the notion of local perturbations

$$\Omega_\rho^{W_\Lambda} = \exp[-\tfrac{1}{2}\beta(H - W_\Lambda)]\,\Omega_\rho$$

of the cyclic vector Ω_ρ in terms of a Dyson expansion, and then prove Theorem 1.3.

Section 4 is devoted to the proof of Theorems 1.4 and 1.5. We use a modified version of Araki's relative entropy method.[1,3,7] Since the quasilocal algebra is a mixture of classical and quantum observables, and since the system is an unbounded spin system, we have to modify the standard methods,[1,2,8] and use several limiting process.

In Section 5, we discuss some open problems for the full quantum model (1.1) and also for ground states ($\beta = \infty$).

2. CONSTRUCTION OF INFINITE-VOLUME-LIMIT THEORY

In this section we obtain useful uniform estimates for finite-volume Gibbs states, and then we construct an infinite-volume-limit theory. At the end of this section we will prove Theorem 1.1.

We first derive uniform bounds for the model. Let $\|h\|_p$ be the l_p-norms defined in (1.18) for any finite sequence $h: Z + 1/2 \to R$. We then have the following bounds:

Proposition 2.1. Let $c = \max\{1/2\beta w^2, 2\beta g/w^2\}$. Then for any $h: Z + 1/2 \to C$

$$|\rho_\Lambda(\exp[\phi(h)])| \leqslant \exp\{c(\|h\|_2^2 + \|h\|_1)\}$$

uniformly in Λ.

Proof. Since $|\rho_\Lambda(\exp[\phi(h)])| \leqslant \rho_\Lambda(\exp[\phi(Reh)])$, it suffices to show the proposition for real h. Recall the definitions of $H_{\Lambda,\mathrm{B}}$, $H_{\Lambda,\mathrm{F}}$, and $\rho_\Lambda(A)$ in (1.4), (1.5), and (1.10), respectively. By changes of variables $\phi_{x+1/2} \to \phi_{x+1/2} - (1/w^2)\, h(x+1/2)$, one obtains that for any real h

$$\rho_\Lambda(\exp[\phi(h)]) \leqslant \exp(c\|h\|_2^2)\, Z_\Lambda^{-1} \times \int \cdots \int \prod_{i \in \Lambda} d\phi_{i+1/2}\, e^{-\beta H_{\Lambda,\mathrm{B}}} \operatorname{Tr}_{\mathrm{F}_\Lambda}(e^{-\beta H_{\Lambda,\mathrm{F}} + G_\Lambda}) \tag{2.1}$$

where

$$G_\Lambda = \beta \sum_{x \in \Lambda} [C_x^* C_{x+1} + C_{x+1}^* C_x] \left[\frac{1}{w^2}\, gh\left(x + \frac{1}{2}\right)\right]$$

Thus, if one shows that

$$\operatorname{Tr}_{\mathrm{F}_\Lambda}(e^{-\beta H_{\Lambda,\mathrm{F}} + G_\Lambda}) \leqslant \exp(c\|h\|_1) \operatorname{Tr}_{\mathrm{F}_\Lambda}(e^{-\beta H_{\Lambda,\mathrm{F}}}) \tag{2.2}$$

the proposition will follow from (2.1) and (2.2). The bound (2.2) follows from the Golden–Thompson inequality

$$\operatorname{Tr}(e^{A+B}) \leqslant \operatorname{Tr}(e^A e^B) \leqslant \operatorname{Tr}(e^A) e^{\|B\|} \tag{2.3}$$

for any self-adjoint matrices A and B, and from the fact that $\|C^x\|_{\mathrm{F}} \leqslant 1$. This proves the proposition.

Lemma 2.2. Let h_j, $j = 1, 2, \ldots, m$, be sequences from $\Lambda + 1/2$ to R. Then there is a constant c, independent of Λ, such that

$$\left|\rho_\Lambda\left(\prod_{j=1}^m \phi(h_j)\right)\right| \leqslant c^m (m!)^{1/2} \prod_{j=1}^m (\|h\|_2^2 + \|h\|_1)$$

uniformly in Λ.

Proof. Let

$$f(z_1, z_2, \ldots, z_m) = \rho_\Lambda\left(\prod_{j=1}^m \exp[z_j \phi(h_j)/\sqrt{m}(\|h_j\|_2^2 + \|h_j\|_1)]\right)$$

Then $f(z_1, z_2,..., z_m)$ is analytic in each variable z_j, $j=1,..., m$, separately. From Proposition 2.1, it follows that for $|z_j| \leqslant 1$

$$|f(z_1,..., z_m)| \leqslant e^{dm}$$

for some constant d uniformly in Λ. Thus, one may use the Cauchy integral formula (on the unit circle C) to conclude that

$$\begin{aligned}
&\left|\rho_\Lambda\left(\prod_{j=1}^m [\phi(h_j)/\sqrt{m}(\|h\|_2^2 + \|h_1\|)]\right)\right| \\
&\quad = \left|\left[\left(\prod_{j=1}^m \frac{\partial}{\partial z_j}\right) f(z_1,..., z_m)\right]_{z_j=0}\right| \\
&\quad = \left|(2\pi)^m \int_C dz_1 \cdots \int_C dz_m f(z_1,..., z_m)/z_1 \cdots z_m\right| \\
&\quad \leqslant e^{dm}
\end{aligned}$$

Since $m^m \leqslant c_1^m m!$ for some constant c_1, the lemma follows from the above bound.

We next establish some commutator estimates. Let $A_\Lambda = A_\Lambda(\phi)$ be a $|\Lambda| \times |\Lambda|$ Hermitian matrix whose ij element is given by

$$A_\Lambda(\phi)_{ij} = \begin{cases} 0, & |i-j| \neq 1 \\ (t + g\phi_{i+1/2}), & |i-j| = 1, \quad i<j \end{cases} \tag{2.4}$$

For given operators A and B we write

$$\delta_A^0(B) = B, \qquad \delta_A^m(B) = [A, \delta_A^{m-1}(B)] \tag{2.5}$$

We also write

$$C(g) = \sum_{i \in Z} C_i g(i), \qquad C^*(g) = \sum_{i \in Z} C_i^* g(i) \tag{2.6}$$

for any finite sequence $g: Z \to R$. For $\Lambda = \{n, n+1,..., m\} \subset Z$ we have

$$H_{\Lambda,\mathrm{F}} = \sum_{i,j \in \Lambda} C_i^* A_\Lambda(\phi)_{ij} C_j \tag{2.7}$$

Here we have used the definitions in (1.5) and (2.4) to derive the above. A direct computation yields

$$\delta_{H_{\Lambda,\mathrm{F}}}^m(C(g)) = C(A_\Lambda(\phi)^m g) \tag{2.8}$$

Since

$$\frac{d^m}{dt^m}\alpha_t^\Lambda(C(g))=i^m\alpha_t^\Lambda(\delta^m_{H_\Lambda,\mathrm{F}}(C(g))) \tag{2.9}$$

it follows from (2.8) that

$$\alpha_t^\Lambda(C(g))=C(e^{itA_\Lambda(\phi)}g), \qquad \alpha_t^\Lambda(C^*(g))=C^*(e^{-itA_\Lambda(\phi)}g) \tag{2.10}$$

We have the following result:

Proposition 2.3. For a fixed $\Lambda'\subset\Lambda$, and for $A\in\mathscr{A}^{\mathrm{F}}_{\Lambda'}$, there is a constant C_A independent of Λ such that

$$\rho_\Lambda((\delta^m_{H_\Lambda,\mathrm{F}}(A))^*\,\delta^m_{H_\Lambda,\mathrm{F}}(A))\leqslant C_A^m m!$$

uniformly in Λ.

Proof. Let $A\in\mathscr{A}^{\mathrm{F}}_{\Lambda'}$. Then A can be identified as an polynomial of C_i and C_i^*, $i\in\Lambda'$, with a degree at most $2^{|\Lambda'|}$. Thus it is sufficient to show the proposition for a monomial of the form

$$A=\prod_{l=1}^{p}\tilde{C}_{i_l}, \qquad i_l\in\Lambda'$$

where $\tilde{C}_i$ is either C_i or else C_i^*. Notice that

$$\delta^m_{H_\Lambda,\mathrm{F}}(A)=\sum_{\substack{m_l=0,1,\ldots,m;\\ \sum m_l=m}}\prod_{l=1}^{m}\delta^{m_l}_{H_\Lambda,\mathrm{F}}(\tilde{C}_{i_l}) \tag{2.11}$$

and that by (2.8)

$$\delta^{m_l}_{H_\Lambda,\mathrm{F}}(C_{i_l})=\pm\tilde{C}(A_\Lambda(\phi)^{m_l}e_{i_l}) \tag{2.12}$$

where e_i is the function defined by $e_i(j)=\delta_{ij}$. Let W_i^m be the family of random walks (in Λ) of m steps starting at the site $i\in\Lambda'$. For $w\in W_i^m$, we denote the final site of w by w_f. Then, from the definition of $A_\Lambda(\phi)$ it follows that

$$\tilde{C}(A_\Lambda(\phi)^m e_i)=\sum_{w\in W_i^m}\left[\prod_{(l_1,l_2)\in w}(t+g\phi_{(l_1+l_2)/2})\right]\tilde{C}_{w_f} \tag{2.13}$$

We note that the total number of terms in (2.11) is p^m and that $\mathrm{card}(W_i^m)\leqslant 2^m$. We substitute (2.12) and (2.13) into (2.11). Then it becomes clear that

$$\rho_\Lambda((\delta^m_{H_\Lambda,\mathrm{F}}(A))^*(\delta^m_{H_\Lambda,\mathrm{F}}(A)))$$

is the sum of $(p^m 2^m)^2$ terms of the form

$$\rho_A\left(\prod_{l=1}^{2m}(t+g\phi_{i_l+1/2})\prod_{k=1}^{2p}\tilde{C}_{i_k}\right)$$

Using Proposition 2.1 and the fact that $\|\tilde{C}_i\|=1$, we bound the above expression by $C^{2m}(2m!)^{1/2}$. This proves the proposition completely.

We will use the following lemma to prove Theorem 1.1.

Lemma 2.4. [Lemma 6.3.23 of Ref. 1]. Let $\{f_\alpha\}$ be a net of $n\geqslant 1$ times continuously differentiable functions from R to C, and assume that f_α converges pointwuse to a function f. Assume that the derivatives of f_α up to order n are bounded on compacts, uniformly in α. It follows that f is $n-1$ times continuously differentiable, and

$$f_\alpha^{(m)}\to f^{(m)}$$

for $m=0, 1,..., n-1$, where the convergence is uniform on compacts.

Using Proposition 2.3 and Lemma 2.4, we obtain the following result:

Proposition 2.5. Let $\{A_\alpha\}$ be a subnet such that

$$G(A, B; t)=\lim_{A_\alpha\to Z} G_{A_\alpha}(A, B; t)$$

exists for $A, B\in\mathscr{A}$ and $t\in R$. For any $A\in\mathscr{A}$ and $B\in\mathscr{A}_A$, $G(A, B; t)$ can be extended to an analytic function $G(A, B; z)$ on the strip $D=\{z: 0<\operatorname{Im} z<\beta\}$, and for any $m\geqslant 0$

$$\frac{d^m}{dz^m}G(A, B; z)=\lim_{A_\alpha\to Z}\frac{d^m}{dz^m}G_{A_\alpha}(A, B; z)$$

on $z\in\bar{D}_\beta$. Furthermore, there is a constant C_B such that

$$\left|\frac{d^m}{dz^m}G(A, B; z)\right|\leqslant\|A\|\, C_B^m(m!)^{1/2}$$

on $z\in\bar{D}_\beta$.

Proof. Since $\mathscr{A}_{A'}^B$ is an Abelian algebra, it is sufficient to show the proposition for $B\in\mathscr{A}_{A'}^{\mathrm{F}}$ for some finite $A'\subset Z$. For given $A\in\mathscr{A}$ and $B\in\mathscr{A}_{A'}^{\mathrm{F}}$ we write

$$f(z)=G_{A_\alpha}(A, B; z)=\rho_{A_\alpha}(A\alpha_z^{A_\alpha}(B)) \tag{2.14}$$

where

$$\alpha_z^A(B)=e^{izH_{A,\mathrm{F}}}Be^{-izH_{A,\mathrm{F}}} \tag{2.15}$$

It then follows that for $z \in D_\beta$

$$\begin{aligned} f_\alpha^{(m)}(z) &= i^m \rho_{\Lambda_\alpha}(A\alpha_z^{\Lambda_\alpha}(\delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)) \\ &= i^m \rho_{\Lambda_\alpha}(\alpha_{-z}^{\Lambda_\alpha}(A)\, \delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)) \end{aligned} \tag{2.16}$$

For a finite Λ_α, the above is analytic in the region D_β. Thus, $|f_\alpha^m(z)|$ has the maximum on the boundary of D_β. But

$$\begin{aligned} |f_\alpha^{(m)}(z)|^2_{z=t} &\leqslant \|A\|^2\, \rho_{\Lambda_\alpha}([\delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)]^*\, \delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)) \\ &\leqslant \|A\|^2\, C_{\mathrm{B}}^m m! \end{aligned}$$

by Proposition 2.3. On the other hand,

$$\begin{aligned} |f_\alpha^m(z)|^2_{z=t+i\beta} &= |\rho_{\Lambda_\alpha}(\delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)\, \alpha_{-t}^{\Lambda_\alpha}(A)| \\ &\leqslant \rho_{\Lambda_\alpha}(\delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)[\delta^m_{H_{\Lambda_\alpha},\mathrm{F}}(B)]^*)\, \rho_{\Lambda_\alpha}(A^*A) \\ &\leqslant \|A\|^2 C_{\mathrm{B}}^m m! \end{aligned}$$

Here we have used the definition of ρ_Λ in (1.10), the KMS conditions for ρ_Λ, the Schwarz inequality for the state ρ_Λ, and Proposition 2.3. Thus, we conclude that

$$|f_\alpha^{(m)}(z)| \leqslant \|A\|\, C_{\mathrm{B}}^m (m!)^{1/2} \tag{2.17}$$

on $\bar{D}_\beta$ uniformly in Λ_α. Thus, $\{f_\alpha(z)\}$ is a net of analytic functions on D_β and each $f_\alpha^{(m)}(z)$ is bounded uniformly by (2.17). One may choose a subnet $\{\Lambda_{\alpha'}\}$ of $\{\Lambda_\alpha\}$ such that $f_{\alpha'}(z) \to f(z)$ on $\bar{D}_\beta$. From (2.17) and Vitali's theorem it then follows that $f_{\alpha'}^{(m)} \to f^{(m)}$ on D_β (and also on the boundary, by Lemma 2.4). The boundedness in the proposition follows from (2.17).

We are now ready to show Theorem 1.1.

Proof of Theorem 1.1. Let $(\mathscr{H}_\rho, \pi_\rho, \Omega_\rho)$ be the cyclic representation with respect to a weak*-limit state ρ of $\{\rho_\Lambda\}$. Then ρ is entire analytic by Proposition 2.5. We define an operator on $\pi_\rho(\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda)\Omega_\rho$ by

$$H\pi_\rho(B)\Omega_\rho = [\pi_\rho(H_{\Lambda,\mathrm{F}} + W_\Lambda), \pi_\rho(B)]\Omega_\rho \tag{2.18}$$

for any $B \in \mathscr{A}_\Lambda$. From Proposition 2.5, it follows that for any $A \in \mathscr{A}$ and $B \in \mathscr{A}_\Lambda$,

$$\begin{aligned} G^{(1)}(A, B; t=0) &= \lim_{\Lambda_\alpha \to Z} G^{(1)}_{\Lambda_\alpha}(A, B; t=0) \\ &= i \lim_{\Lambda_{\alpha'} \to Z} \rho_{\Lambda_{\alpha'}}(A\delta_{H_\Lambda + W_\Lambda}(B)) \\ &= i(\pi_\rho(A)^*\Omega_\rho, \pi_\rho(\delta_{H_\Lambda + W_\Lambda}(B))\Omega_\rho) \\ &= i(\pi_\rho(A)^*\Omega_\rho, H\pi_\rho(B)\Omega_\rho) \end{aligned} \tag{2.19}$$

Similarly, it follows from Proposition 2.5 that for any $B \in \mathscr{A}_\Lambda$,

$$G^{(m)}(A, B; t=0) = i^m(\pi_\rho(A^*)\Omega_\rho, H^m\pi_\rho(B)\Omega_\rho) \tag{2.20}$$

and

$$|(\pi_\rho(A^*)\Omega_\rho, H^m\pi_\rho(B)\Omega_\rho)| \leqslant \|A\| \, C_B^m(m!)^{1/2} \tag{2.21}$$

Since $\pi_\rho(\mathscr{A})\Omega_\rho$ is dense, one may conclude that $\pi_\rho(B)\Omega_\rho$ is an analytic vector for H. Notice that $H\Omega_\rho = 0$. Since $\pi_\rho(\bigcup_{\Lambda \subset Z} \mathscr{A}_\Lambda)\Omega_\rho$ is dense, H is essentially self-adjoint on $\pi_\rho(\bigcup \mathscr{A}_\Lambda)\Omega_\rho$ and

$$G(A, B; t) = (\pi_\rho(A^*)\Omega_\rho, e^{itH}\pi_\rho(B)e^{-itH}\Omega_\rho)$$

Since $H\Omega_\rho = 0$ by (2.18), Ω_ρ is invariant under e^{itH}. This proves Theorem 1.1 completely.

3. GIBBS CONDITIONS FOR INFINITE-VOLUME-LIMIT STATES

This section is devoted to the proof of Theorem 1.3. For notational simplification we make the following convention. *In the rest of the paper we suppress the representation notation* π_ρ if there is no confusion involved. Thus, we will use the abbreviated notation A for $\pi_\rho(A)$.

We note that any weak*-limit ρ of $\{\rho_\Lambda\}$ satisfies the following weak KMS conditions: For any $A, B \in \mathscr{A}$ and $\hat{f} \in \mathscr{D}$,

$$\int dt\, f(t)\, G(A, B; t) = \int dt\, f(t + i\beta)\, G(B, A; -t) \tag{3.1}$$

The above equality follows from the corresponding KMS conditions for the states ρ_Λ and Proposition 2.5. The above relation implies that Ω_ρ is a separating vector for $\pi_\rho(\mathscr{A})''$ (Example 5.3.13 of Ref. 1). The regularity condition (G-1) follows from Lemma 2.1 and Proposition 2.5:

Lemma 3.1. Any weak*-limit ρ of $\{\rho_\Lambda\}$ satisfies conditions (G-1) and (G-2) in Definition 1.2.

Let H be the self-adjoint operator in Theorem 1.1, and let

$$\alpha_t(A) = e^{itH}Ae^{-itH} \tag{3.2}$$

for any $A \in \mathscr{L}(\mathscr{H}_\rho)$. Our next task is to identify the modular automorphism by α_t. For a given A, the operator $H_{\Lambda,\mathrm{F}}$ $[=\pi_\rho(H_{\Lambda,\mathrm{F}})]$ is essentially self-adjoint on $\pi_\rho(\bigcup \mathscr{A}_\Lambda)$ by the regularity of ρ. Let

$$\alpha_{t,\Lambda'}(A) = [\exp(itH_{\Lambda',\mathrm{F}})]\, A[\exp(-itH_{\Lambda',\mathrm{F}})] \tag{3.3}$$

We then have the following result:

Lemma 3.2. For any $A \in \pi_\rho(\mathscr{A}_\Lambda)$ and $B \in \pi_\rho(\bigcup \mathscr{A}_\Lambda)$,

$$\alpha_{t,\Lambda}(A)\, B\Omega_\rho \to \alpha_t(A)\, B\Omega_\rho$$

strongly.

Proof. From (2.18) it follows that for any $A \in \pi_\rho(\mathscr{A}_\Lambda)$ and $B \in \pi_\rho(\mathscr{A}_\Lambda)$,

$$[H, A]\, B\Omega_\rho = [(H_{\Lambda,\mathrm{F}} + W_\Lambda), A]\, B\Omega_\rho \tag{3.4}$$

Let $\Lambda^{(m)} = \{x \in Z\colon \operatorname{dist}(x, \Lambda) \leqslant m\}$. Then

$$H_{\Lambda,\mathrm{F}} + W_\Lambda = H_{\Lambda^{(1)},\mathrm{F}}$$

Using (3.4) m times, we obtain that for $A \in \pi_\rho(\mathscr{A}_\Lambda)$,

$$\delta^m_H(A)\, B\Omega_\rho = \delta^m_{H_{\Lambda^{(m)},\mathrm{F}}}(A)\, B\Omega_\rho \tag{3.5}$$

Notice that it suffices to prove the lemma for $A \in \in \pi_\rho(\mathscr{A}^{\mathrm{F}}_\Lambda)$, and so for $A = \prod_{j=1}^{p} \tilde{C}_j$, $i \in \Lambda$, $p \leqslant |\Lambda|$. We expand (3.2) and (3.3) in power series. Then for $B \in \pi_\rho(\mathscr{A})$,

$$\begin{aligned}
[\alpha_i(A) - \alpha_{t,\Lambda^{(m)}}(A)]\, B\Omega &= (it)^n \sum_{n=1}^{\infty} [\delta^n_H(A) - \delta^n_{H_{\Lambda^{(m)},\mathrm{F}}}(A)]\, B\Omega \\
&= \frac{1}{n!} (it)^n \sum_{n \geqslant m} [\delta^n_{H_{\Lambda^{(n)},\mathrm{F}}}(A) - \delta^n_{H_{\Lambda^{(m)},\mathrm{F}}}(A)]\, B\Omega
\end{aligned} \tag{3.6}$$

where we have used (3.5) to obtain the second equality. Using a method similar to that used in the proof of Proposition 2.3, one may show that the norm of the nth term in (3.6) is bounded by $C^n_\Lambda (n!)^{1/2}$ and so (3.6) tends to zero as $m \to \infty$. This prove the lemma.

Proposition 3.3. Let α be defined as in (3.2). Then α is an automorphism on $\pi(\mathscr{A})''$.

Proof. Since $\pi_\rho(\bigcup \mathscr{A}_\Lambda)$ is dense in $\pi_\rho(\mathscr{A})$, for any $B \in \pi_\rho(\mathscr{A})''$, there exists a sequence $\{B_n\} \subset \pi_\rho(\bigcup \mathscr{A}_\Lambda)$ such that $B_n \to B$ weakly and so $\alpha_t(B_n) \to \alpha_t(B)$ weakly. Since $\alpha_t(B_n) \in \pi_\rho(\mathscr{A})''$ by Lemma 3.2, it follows that $\alpha_t(B) \in \pi(\mathscr{A})''$.

Corollary 3.4. Let h be the generator of the modular automorphism on $\pi(\mathscr{A})''$. Then $h = H$.

Proof. From the KMS conditions for any finite-volume Gibbs states ρ_Λ and from Proposition 2.5 one may deduce that $\rho(A\alpha_t(B))$ satisfies the

KMS condition for $A \in \mathscr{A}$ and $B \in \mathscr{A}_\Lambda$. Since $\bigcup \mathscr{A}_\Lambda$ is norm dense, $\rho(A\alpha_t(B))$ satisfies the KMS condition for $A, B \in \mathscr{A}$. By Corollary 5.3.4 of Ref. 2, $(\pi(\mathscr{A}), \alpha_t)$ satisfies the KMS condition. The result follows from the uniqueness of the modular automorphism (Theorem 5.3.10 of Ref. 2).

In the rest of this section we prove the Gibbs conditions (G-4) in Definition 1.2 for ρ. We first give a meaning to $\exp[-\frac{1}{2}\beta(H-W_\Lambda)]\Omega_\rho$. A direct calculation shows that W_Λ is essentially self-adjoint on $\pi_\rho(\bigcup \mathscr{A}_\Lambda)$. Thus, it may be possible to show that $H - W_\Lambda$ is a self-adjoint operator on a suitable dense domain including $\pi_\rho(\bigcup \mathscr{A}_\Lambda)$. Instead, we will use a Dyson expansion. Let Γ_t^Λ be a one-parameter family of elements given by

$$\Gamma_t^\Lambda = 1 + \sum_{n \geqslant 1} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, \alpha_{t_n}(W_\Lambda) \cdots \alpha_{t_1}(W_\Lambda) \qquad (3.7)$$

where $\alpha_t(A) = e^{itH} A e^{-itH}$. The above expression is formally equivalent to[2]

$$\Gamma_t^\Lambda = e^{it(H-W_\Lambda)} e^{-itH}$$

Proposition 3.5. (1) For any finite $\Lambda \subset Z$, the vector-valued functions

$$P_\Lambda^{(n)}(z)\Omega_\rho = e^{iz_n H} W_\Lambda e^{i(z_{n-1}-z_n)H} W_\Lambda \cdots W_\Lambda e^{-iz_1 H} \Omega_\rho$$

are analytic in the tube $D_{\beta/2}^{(n)}$ defined by

$$D_\alpha^{(n)} = \{ z \in C^n \colon \; 0 < \operatorname{Im} z_n < \operatorname{Im} z_{n-1} < \cdots < \operatorname{Im} z_1 < \infty \}$$

The function $P_\Lambda^{(n)}(z)\Omega_\rho$ is strongly continuous and uniformly bounded on its closure $\overline{D_{\beta/2}^{(n)}}$, and

$$\sup_{z \in D_{\beta/2}^{(n)}} \| P_\Lambda^{(n)}(z)\Omega_\rho \| \leqslant C^n (n!)^{1/2}$$

uniformly in Λ.

(2) The series

$$\Gamma_{i\beta/2}^\Lambda \Omega_\rho = \Omega_\rho + \sum_{n \geqslant 1} \int_{0 < s_n < \cdots < s_1 < \beta/2} ds_1 \cdots ds_n \, \alpha_{is_n}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\rho$$

converges strongly.

Proof. We note that be Lemma 2.2

$$\left\| \prod_{i=1}^n (|t + g\phi_{l_i+1/2}|)\Omega_\rho \right\| \leqslant C^n (n!)^{1/2}$$

and so

$$\|(W_\Lambda)^n \Omega_\rho\| \leqslant C^n (n!)^{1/2} \qquad (3.8)$$

uniformly in Λ. We note that, as a consequence of Lemma 3.1 and Corollary 3.4, ρ is an α-KMS state on $\pi(\mathscr{A})''$. Now the proposition follows from the bound (3.8), the KMS conditions on ρ, and a method similar to that used in the proof of Theorem 5.4.4 of Ref. 2. For the details we refer to Ref. 2.

Remark. Formally one may see that

$$\Gamma^{\Lambda}_{i\beta/2}\Omega_{\omega} = \exp[-\tfrac{1}{2}\beta(H - W_{\Lambda})]\,\Omega_{w}$$

Thus the condition (G-4) in Definition 1.2 makes sense.

We now complete the proof of Theorem 1.3 by proving the condition (G-4) in Definition 1.2:

Proof of Theorem 1.3. By Proposition 3.3, Corollary 3.4, and (2.13) we only need to show that the condition (G-4) in Definition 1.2 holds for ρ. Let Λ and Λ' be finite subsets of Z such that $\Lambda \subset \Lambda' \subset Z$, and let

$$\alpha^{\Lambda'}_{z}(A) = [\exp(izH_{\Lambda',\mathrm{F}})]\,A\,[\exp(-izH_{\Lambda',\mathrm{F}})] \tag{3.9}$$

for $A \in \mathscr{A}_{\Lambda}$. It then follows from a Dyson expansion that

$$\begin{aligned} \Gamma^{\Lambda,\Lambda'}_{t} &\equiv \exp[it(H_{\Lambda',\mathrm{F}} - W_{\Lambda})]\,\exp(itH_{\Lambda',\mathrm{F}}) \\ &= 1 + \sum_{n\geqslant 1} (-i)^{n} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \cdots \int_{0}^{t_{n-1}} dt_{n}\, \alpha^{\Lambda'}_{t_{n}}(W_{\Lambda}) \cdots \alpha^{\Lambda'}_{t_{1}}(W_{\Lambda}) \end{aligned}$$

We define

$$\begin{aligned} P^{(n)}_{\Lambda,\Lambda'}(z) &= \alpha^{\Lambda'}_{z_{n}}(W_{\Lambda}) \cdots \alpha^{\Lambda'}_{z_{1}}(W_{\Lambda}) \\ Q^{(m)}_{\Lambda,\Lambda'}(\xi) &= \alpha^{\Lambda'}_{\xi_{1}}(W_{\Lambda}) \cdots \alpha^{\Lambda'}_{\xi_{m}}(W_{\Lambda}) \end{aligned} \tag{3.10}$$

and for $B \in \mathscr{A}_{\Lambda''}$, $\Lambda'' \subset \Lambda'$,

$$F^{(m,n)}_{\Lambda,\Lambda'}(B; \xi, z) = \rho_{\Lambda'}(Q^{(m)}_{\Lambda,\Lambda'}(\xi)\, B P^{(n)}_{\Lambda,\Lambda'}(z)) \tag{3.11}$$

Then by the KMS conditions for $\rho_{\Lambda'}$ the regularity of ρ_{Λ} (Lemma 2.2), and a method similar to that used in the proof of Theorem 5.4.4 of Ref. 2, it is easy to show that the functions $F^{(m,n)}_{\Lambda,\Lambda'}(B; \xi, z)$ are analytic on the domain

$$\begin{aligned} D^{(m,n)}_{\beta/2} = \{ &(\xi, z) \in C^{m+n}: \\ &-\tfrac{1}{2}\beta < \operatorname{Im} \xi_{1} < \cdots < \operatorname{Im} \xi_{m} < 0 < \operatorname{Im} z_{n} < \cdots < \operatorname{Im} z_{1} < \tfrac{1}{2}\beta \} \end{aligned} \tag{3.12}$$

continuous on its closure and bounded uniformly in Λ' on its closure by

$$|F^{(m,n)}_{\Lambda,\Lambda'}(B; \xi, z)| \leqslant C^{m+n}(m!)^{1/2}(n!)^{1/2} \tag{3.13}$$

Moreover, using a method similar to that used in the proof of Proposition 2.5 (and Proposition 2.3), one may prove that any derivative of $F_{\Lambda,\Lambda'}^{(m,n)}(B; \xi, z)$ is also bounded uniformly in Λ'. Let $\{\Lambda'_\alpha\}$ be the subnet of $\{\Lambda'\}$ such that $G_{\Lambda'_\alpha}(A, B; t)$ converges to $G(A, B; t)$. Using either Lemma 2.4 or else the method on p. 408 of Ref. 2 (with Vitali's theorem), one may choose a subnet $\Lambda'_{\alpha'}$ of Λ'_α such that

$$F_{\Lambda}^{(m,n)}(B; \xi, z) = \lim_{\Lambda'_{\alpha'} \to Z} F_{\Lambda,\Lambda'_{\alpha'}}(B; \xi, z)$$

on the closure of $D_{\beta/2}^{(m,n)}$ (uniformly in compacts). Thus, we have

$$\begin{aligned}(\Omega_\rho, \alpha_{is'_1}(W_\Lambda) \cdots \alpha_{is'_m}(W_\Lambda)\, B\alpha_{is_n}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\rho) \\ = \lim_{\Lambda'_{\alpha'} \to Z} \rho_{\Lambda'_\alpha}(\alpha_{is'_1}^{\Lambda'_\alpha}(W_\Lambda) \cdots \alpha_{is'_m}^{\Lambda'_\alpha}(W_\Lambda)\, B\alpha_{is_n}^{\Lambda'_\alpha}(W_\Lambda) \cdots \alpha_{is_1}^{\Lambda'_\alpha}(W_1)) \quad (3.14)\end{aligned}$$

for any $-\frac{1}{2}\beta < s'_1 < \cdots < s'_m < 0$ and $0 < s_n < \cdots < \frac{1}{2}\beta$. Integrating both sides of (3.14) over the regions, summing over m and n, and using the uniform bounds in (3.14) (together with Proposition 3.5), we obtain the following result: Let

$$\Gamma_{i\beta/2}^{\Lambda,\Lambda'} = \exp[-\tfrac{1}{2}\beta(H_{\Lambda',\mathrm{F}} - W_\Lambda)]\, \exp(\tfrac{1}{2}\beta H_{\Lambda',\mathrm{F}})$$

Then for any $A \in \mathscr{A}_\Lambda$, $B \in \mathscr{A}_{\Lambda'}$, $\Lambda' \subset \Lambda^c$,

$$\begin{aligned}&(\Gamma_{i\beta/2}^{\Lambda}\Omega_\omega, \pi_\rho(A)\, \pi_\rho(B)\, \Gamma_{i\beta/2}^{\Lambda}\Omega_\rho) \\ &\quad = \lim_{\Lambda'_{\alpha'} \to Z} \rho_{\Lambda'_{\alpha'}}((\Gamma_{i\beta/2}^{\Lambda,\Lambda'_{\alpha'}})^* AB\Gamma_{i\beta/2}^{\Lambda,\Lambda'_{\alpha'}}) \\ &\quad = \lim_{\Lambda'_{\alpha'} \to Z} \rho_{\Lambda'_{\alpha'}}(A)\, \rho_{\Lambda'_{\alpha'}}((\Gamma_{i\beta/2}^{\Lambda,\Lambda'_{\alpha'}})^* B\Gamma_{i\beta/2}^{\Lambda,\Lambda'_{\alpha'}}) \\ &\quad = \rho(A)(\Gamma_{i\beta/2}^{\Lambda}\Omega_\rho, \pi_\rho(B)\, \Gamma_{i\beta/2}^{\Lambda}\Omega_\rho)\end{aligned}$$

Here we have used the fact that for any $A \in \mathscr{A}_\Lambda^{\mathrm{F}}$, $B \in \mathscr{A}_{\Lambda'}^{\mathrm{F}}$, $\Lambda \cap \Lambda' = \varnothing$,

$$\mathrm{Tr}_{F_{\Lambda \cup \Lambda'}}(AB) = \mathrm{const} \times \mathrm{Tr}_{F_\Lambda}(A)\, \mathrm{Tr}_{F_{\Lambda'}}(B)$$

to obtain the second inequality in the above expressions. This proves Theorem 1.3 completely.

4. UNIQUENESS OF GIBBS STATES

This section is devoted to proving the uniqueness of Gibbs states. Our main strategy is as follows: Let Γ be the set of Gibbs states on $\mathscr{A}$ (or an algebra $\tilde{\mathscr{A}}$ defined below). We will show that Γ is a metrizable Choquet simplex, and so each state in Γ is the barycenter of a unique probability measure concentrated on extremal points of Γ.[2,6] We then introduce the notion of the algebra at infinity, $\mathscr{B}_\omega$, for $\omega \in \Gamma$. It will follow that, if $\omega \in \Gamma$ is extremal, then $\mathscr{B}_\omega = \{c\mathbb{1}\}$. Finally, let ω be an extremal Gibbs state and let ρ be a weak*-limit state of finite-volume Gibbs states ρ_Λ. Using the fact that expectations of the surface energies W_Λ are bounded uniformly in Λ, and using the Gibs conditions for $\omega \in \Gamma$, we show that there is a positive operator T affiliated with $\mathscr{B}_\omega$ such that $\omega(TA) = \rho(A)$. Since $\mathscr{B}_\omega = \{c\mathbb{1}\}$, this will prove Theorem 1.4.

Our method is closely related to those in Refs. 1, 3, and 7. But, as stated in the introduction, we need to modify the methods in Refs. 1 and 3 in order to take care of unbounded spins and the mixture of classical and quantum observables. This makes it necessary to take several limiting processes in the proof.

Before proving Theorem 1.4, we need some preparation. The first question is whether the set Γ of Gibbs states is metrizable in the weak*-topology. Using an argument similar to that used in Ref. 12 and the regularity of $\omega \in \Gamma$, one may be able to show the metrizability directly. Instead, we give the following argument here. Let $\dot{\mathbb{R}}$ be the one-point compactification of $\mathbb{R}$ and let

$$\tilde{\mathscr{A}} = \left(\bigcup_{\Lambda \subset Z} \mathscr{C}(\dot{\mathbb{R}}^{|\Lambda|}) \otimes \mathscr{A}_\Lambda^{\mathrm{F}} \right)^- \tag{4.1}$$

Then $\tilde{\mathscr{A}}$ is separable and so the set $\tilde{\mathscr{E}}$ of states on $\tilde{\mathscr{A}}$ is metrizable in the weak*-topology. Let $\tilde{\omega} \in \tilde{\mathscr{E}}$ be a state satisfying the bound

$$\sum_{m=0}^{\infty} |z|^m \{\tilde{\omega}(\phi_{i+1/2}^m \exp[-\alpha\phi_{i+1/2}^2])\} \leqslant \exp[c(|z|^2 + |z|)] \tag{4.2}$$

uniformly in $\alpha \in (0, \infty)$, where c is the constant given in the condition (G-1) in Definition 1.2. Then $\tilde{\omega}$ defines a regular state on $\mathscr{A}$ uniquely, which we denote again by $\tilde{\omega}$. Furthermore, one has $\pi_{\tilde{\omega}}(\tilde{\mathscr{A}})'' = \pi_{\tilde{\omega}}(\mathscr{A})''$. Let $\tilde{\Gamma}$ be the set of states on $\tilde{\mathscr{A}}$ satisfying the bounds (4.2) and the conditions (G-2)–(G-4) in Definition 1.2. Then $\tilde{\Gamma}$ is metrizable. On the other hand, any regular state ω on $\mathscr{A}$ defines a state on $\tilde{\mathscr{A}}$ satisfying the bounds (4.2). Thus, in order to simplify our arguments, we assume that the set Γ of Gibbs states is metrizable in the weak*-topology. Otherwise, one may replace Γ by $\tilde{\Gamma}$ in the rest of this section, and then our conclusions still hold for $\tilde{\Gamma}$ and $\tilde{\mathscr{A}}$.

We first collect some results, which are consequences of the conditions (G-1)–(G-3) in Defintion 1.2. As before, for $\omega \in \Gamma$ let

$$\begin{aligned} \alpha_z(B) &= e^{izh} B e^{-izh} \\ \alpha_z^\Lambda(B) &= e^{izH_{\Lambda,\mathrm{F}}} B e^{-izH_{\Lambda,\mathrm{F}}} \end{aligned} \tag{4.3}$$

Here we have suppressed the representation notation [i.e., $H_{\Lambda,\mathrm{F}} = \pi_\omega(H_{\Lambda,\mathrm{F}})$]. Then we have the following results:

Proposition 4.1. (1) For any finite $\Lambda \subset Z$ and for any $\omega \in \Gamma$ the vector-valued functions

$$P_\Lambda^{(n)}(z)\Omega_\omega = \alpha_{z_n}(W_\Lambda) \cdots \alpha_{z_1}(W_\Lambda)\Omega_\omega$$

are analytic in the tube

$$D_{\beta/2}^{(n)} = \{z \in \mathbb{C}^n\colon\ 0 < \operatorname{Im} z_n < \cdots < \operatorname{Im} z_1 < \beta/2\}$$

The function $P_\Lambda^{(n)}(z)\Omega_\omega$ is strongly continuous and uniformly bounded on its closure $\overline{D_{\beta/2}^{(n)}}$, and

$$\sup_{z \in D_{\beta/2}^{(n)}} \|P_\Lambda^{(n)}(z)\Omega_\omega\| \leqslant C^n(n!)^{1/2}$$

uniformly in Λ and $\omega \in \Gamma$.

(2) For $\omega \in \Gamma$, the series

$$\Gamma_{i\beta/2}^\Lambda \Omega_\omega = \Omega_\omega + \sum_{n \geqslant 1} \int_{0 \leqslant s_n \leqslant \cdots \leqslant s_1 \leqslant \beta/2} ds_1 \cdots ds_n\, \alpha_{is_n}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\omega$$

converges strongly and uniformly in $\omega \in \Gamma$.

(3) For $\omega \in \Gamma$, $B \in \pi_\omega(\mathscr{A}_\Lambda)$, and $A \in \pi_\omega(\mathscr{A})$,

$$\alpha_z^{\Lambda'}(B)\, A\Omega_\omega \to \alpha_z(B)\, A\Omega_\omega \qquad \text{as} \quad \Lambda' \to Z$$

strongly for $D_{\beta/2}^{(1)}$. The convergence is uniform in $\omega \in \Gamma$.

(4) Let $\omega \in \Gamma$. Then for $0 \leqslant s_n \leqslant \cdots \leqslant s_1 \leqslant \beta/2$,

$$\alpha_{is_n}^{\Lambda'}(W_\Lambda) \cdots \alpha_{is_1}^{\Lambda'}(W_\Lambda)\Omega_\omega \to \alpha_{is_n}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\omega$$

strongly as $\Lambda' \to Z$. The convergence is uniform in $\omega \in \Gamma$.

Remark. For any $\omega \in \Gamma$ the modularity condition (G-2) implies that ω satisfies the *KMS conditions*

$$(\Omega_\omega, A\alpha_{i\beta}(B)\Omega_\omega) = (\Omega_\omega, BA\Omega_\omega) \tag{4.4}$$

for any $A, B \in \pi_\omega(\mathscr{A})''$.

Proof of Proposition 4.1. Parts (1) and (2). In the proof of Proposition 3.5, we used only the KMS conditions and the regularity of ρ

[to obtain (3.8)]. Since ω satisfies the same conditions, the proposition follows from the same method as that used in the proof of Proposition 3.5. The uniformity in $\omega \in \Gamma$ follows from uniform bounds on the right-hand side of (3.8) by the regularity condition (G-1).

Part (3). Let $\Lambda^{(m)} = (j \in Z\colon \operatorname{dist}(j, \Lambda) \leqslant m\}$. Then the condition (G-3) implies that for $B \in \pi_\omega(\mathscr{A}_\Lambda)$, $A \in \pi_\omega(\mathscr{A})$,

$$\delta_h^m(B)\, A\Omega_\omega = \delta^m_{H_{\Lambda^{(m)},\mathrm{F}}}(B)\, A\Omega_\omega \tag{4.5}$$

and so it follows that for $z \in D^{(1)}_{\beta/2}$

$$\begin{aligned}&\|\alpha_z(B)\, A\Omega_\omega - \alpha_z^{\Lambda'}(B)\, A\Omega_\omega\|\\ &\quad\leqslant \sum_{n=n'}^{\infty} \frac{|z|^n}{n!} \|[\delta^n_{H_{\Lambda^{(n)},\mathrm{F}}}(B) - \delta^n_{H_{\Lambda',\mathrm{F}}}(B)]\, A\Omega_\omega\|\end{aligned}$$

where $m' = \operatorname{dist}\{\Lambda, \partial\Lambda'\}$. Using a mthod similar to that used in the proof of Proposition 2.3 and the regularity condition (G-1), one may show that the norm of the nth term in the above expression is bounded by $c^n(n!)^{1/2}$ uniformly in Λ' and $\omega \in \Gamma$. This proves part (3) of the proposition.

Part (4). Notice that

$$\begin{aligned}&\alpha^{\Lambda'}_{is_n}(W) \cdots \alpha^{\Lambda'}_{is_1}(W_\Lambda)\Omega_\omega - \alpha_{is_n}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\omega\\ &\quad= \sum_{k=1}^{n} \alpha^{\Lambda'}_{is_n}(W_\Lambda) \cdots \alpha^{\Lambda'}_{is_{k+1}}(W_\Lambda)[\alpha^{\Lambda'}_{is_k}(W_\Lambda) - \alpha_{is_k}(W_\Lambda)]\\ &\quad\quad\times \alpha_{is_{k-1}}(W_\Lambda) \cdots \alpha_{is_1}(W_\Lambda)\Omega_\omega\end{aligned} \tag{4.6}$$

We expand $\alpha^{\Lambda'}_{is}(W_\Lambda)$ by

$$\alpha^{\Lambda'}_{is}(W_\Lambda) = \sum_{m=0}^{\infty} \frac{s^m}{m!} \delta^m_{H_{\Lambda',\mathrm{F}}}(W_\Lambda) \tag{4.7}$$

and $\alpha_{is}(W_\Lambda)$ by an expansion similar to (4.7). Substitute (4.7) into (4.6) to express (4.6) into the following form:

$$\begin{aligned}&\sum_{k=1}^{n} \sum_{m_1,\ldots,m_{k-1}=1}^{\infty} \sum_{m_{k+1},\ldots,m_n=1}^{\infty} \sum_{m_k=m'}^{\infty} \frac{s^{m_1+\cdots+m_n}}{m_1!\cdots m_n!}\\ &\quad\left[\prod_{j=1}^{k-1} \delta^{m_j}_{H_{\Lambda',\mathrm{F}}}(W_\Lambda)\right][\delta^{m_k}_{H_{\Lambda',\mathrm{F}}}(W_\Lambda) - \delta_h^{m_k}(W_\Lambda)] \prod_{j=k+1}^{n} \delta_h^{m_j}(W_\Lambda)\Omega_\omega\end{aligned} \tag{4.8}$$

We now use (4.5), a method similar to that used in the proof of

Proposition 2.3, and the definition of W_Λ in (1.17) to show that each term in the sum in (4.8) is bounded by the following type:

$$C^{m_1+\cdots+m_n}\omega\left(\prod_{j=1}^{n}\prod_{l=1}^{m_j+1}[(t+g\phi_{jl+1/2})]\right)$$
$$\leqslant C_1^{m_1+\cdots+m_n}((m_1+\cdots+m_n+n)!)^{1/2}$$

Uniformly in Λ, Λ' and $\omega\in\Gamma$. Here we have used the regularity condition (G-1) and the method in the proof of Lemma 2.4 to obtain the above inequality. The convergence follows from (4.8) and the above bound.

Proposition 4.2. The set Γ is convex and compact in the weak*-topology.

Proof. Since the set of states is compact, it suffices to show that Γ is convex and closed. Let $\omega_1, \omega_2\in\Gamma$ and let $\omega=\alpha_1\omega_1+\alpha_2\omega_2$, $\alpha_1+\alpha_2=1$. Obviously ω is regular. Define an operator H on $\pi_\omega(\bigcup\mathscr{A}_\Lambda)\Omega_\omega$ by

$$HA\Omega_\omega=[H_{\Lambda,\mathrm{F}}+W_\Lambda, A]\Omega_\omega, \qquad A\in\pi_\omega(\mathscr{A}_\Lambda)$$

Following the argument in the proofs of Theorem 1.1 and Proposition 4.1(3), one can show that H is essentially self-adjoint on $\pi_\omega(\bigcup\mathscr{A}_\Lambda)\Omega_\omega$ and that α_t: $A\mapsto e^{itH}Ae^{-itH}$ defines an automorphism on $\pi(\mathscr{A})''$. Next we prove that $(\pi(\mathscr{A})'', \alpha)$ satisfies the KMS conditions. Notice that for any $A, B\in\bigcup\mathscr{A}_\Lambda$

$$\begin{aligned}
(\Omega_\omega, A\alpha_{i\beta}(B)\Omega_\omega) &= \lim_{\Lambda'\to Z}(\Omega_\omega, A\alpha_{i\beta}^{\Lambda'}(B)\Omega_\omega)\\
&= \lim_{\Lambda'\to Z}\omega(A\alpha_{i\beta}^{\Lambda'}(B))\\
&= \lim_{\Lambda'\to Z}\left[\sum_{j=1}^{2}\alpha_j\omega_j(A\alpha_{i\beta}^{\Lambda'}(B))\right]\\
&= \sum_{j=1}^{2}\alpha_j(\Omega_{\omega_j}, A\alpha_{i\beta}(B)\Omega_{\omega_j})\\
&= \sum_{j=1}^{2}\alpha_j(\Omega_{\omega_j}, BA\Omega^{\omega_j})\\
&= \sum_{j=1}^{2}\alpha_j\omega(BA)\\
&= \omega(BA)\\
&= (\Omega_\omega, BA\Omega_\omega)
\end{aligned}$$

Here we have used Proposition 4.1(3) and the modularity of ω_1 and ω_2. Since $\pi_\omega(\bigcup \mathscr{A}_\Lambda)$ is dense in $\pi(\mathscr{A})''$, Ω_ω is separating for $\pi(\mathscr{A})''$. Thus, ω satisfies $(\pi(\mathscr{A})'', \alpha)$-KMS conditions. By uniqueness of the modular automorphism, we conclude that $H=h$. This prove the conditions (G-2) and (G-3) for ω.

Next we prove the condition (G-4). Using the expansion in Proposition 4.1(2) for $\Omega_\omega^\Lambda \equiv \Gamma_{i\beta/2}^\Lambda \Omega_\omega$ and using Proposition 4.1(4), it is easy to check that for any $A \in \mathscr{A}_\Lambda$, $B \in \mathscr{A}_{\Lambda'}$, $\Lambda' \subset \Lambda^c$,

$$\begin{aligned}(\Omega_\omega^\Lambda, AB\Omega_\omega^\Lambda) &= \sum_{i=1,2} \alpha_i(\Omega_{\omega_i}^\Lambda, AB\Omega_{\omega_i}^\Lambda)\\ &= \rho_\Lambda(A) \sum_{i=1,2} \alpha_i(\Omega_{\omega_i}^\Lambda, B\Omega_{\omega_i}^\Lambda)\\ &= \rho_\Lambda(A)(\Omega_\omega^\Lambda, B\Omega_\omega^\Lambda)\end{aligned}$$

To obtain the first and third inequalities, we have used Proposition 4.1(2) and 4.1(4). Thus, Γ is convex.

Let $\{\omega_j\}$ be a sequence in Γ convergent to ω in the weak*-topology. Obviously ω is regular. Using the method in the first part of this proof and using Proposition 4.1(3), it is easy to show that ω satisfies conditions (G-2)–(G-3). Recall Proposition 4.1(2). We use Proposition 4.1(2) and 4.1(3) and the Gibbs conditions for ω_j to show that ω satisfies condition (G-4). In this proof, the uniform convergence in $\omega \in \Gamma$ in Proposition 4.1(2) and 4.1(3) is in need. We leave the detailed proof to the reader. Thus, Γ is closed.

We investigate the vector $\Omega_\omega^\Lambda = \Gamma_{i\beta/2}^\Lambda \Omega_\omega$ for a given $\omega \in \Gamma$ in more detail.

Lemma 4.3. For each finite $\Lambda \subset Z$ and $\omega \in \Gamma$, the vector $\Omega_\omega^\Lambda = \Gamma_{i\beta/2}^\Lambda \Omega_\omega$ is cyclic and separating for $\pi(\mathscr{A})''$.

Proof. Let $W_{\Lambda,n}$ be the approximate surface energy obtained from W_Λ by replacing $\phi_{i+1/2}$ by $\phi_{i+1/2}[1+(1/n)\phi_{i+1/2}^2]^{-1}$. Then the operator $W_{\Lambda,n}$ is bounded. Let

$$\Gamma_z^{\Lambda,n} = e^{iz(h - W_{\Lambda,n})} e^{-izh} \tag{4.9}$$

Then, by Theorem 5.4.4 (and Corollary 5.4.5) of Ref. 2, $\Omega_\omega^{\Lambda,n} = \Gamma_{i\beta/2}^{\Lambda,n} \Omega_\omega$ is cyclic and separating for $\pi(\mathscr{A})''$. Furthermore, $\Omega_\omega^{\Lambda,n}$ satisfies KMS conditions for

$$\alpha_t^{W_{\Lambda,n}}(A) = e^{it(h - W_{\Lambda,n})} A e^{-it(h - W_{\Lambda,n})} \tag{4.10}$$

Let V_ω be the (closed) natural cone corresponding to $(\pi(\mathscr{A})'', \Omega_\omega)$ (see Section 2.5.4 of Ref. 2). Then $\Omega_\omega^{\Lambda,n}/\|\Omega_\omega^{\Lambda,n}\|$ is the unique normalized representation of the $\alpha^{W_{A,\alpha}}$-KMS state $\omega^{A,n}$ contained in the cone V_ω (p. 160 of Ref. 2).

We define an operator H^A $(=h-W_A)$ on the Hilbert space $\mathscr{H}^A=\overline{\pi_\omega(\mathscr{A})\Omega_\omega^A}$ by

$$H^A A\Omega_\omega^A=[h-W_A, A]\Omega_\omega^A \tag{4.11}$$

Using Proposition 5.1(2), the regularity of $\omega\in\Gamma$, and a method similar to that used in Proposition 3.5, one may obtain that

$$\|(\phi_{i+1/2})^m\Omega_\omega^A\|\leqslant c^m(n!)^{1/2} \tag{4.12}$$

We use the above bounds and the method employed in the proof of Theorem 1.1 to conclude that H^A is essentially self-adjoint on $\bigcup\pi_\omega(\mathscr{A}_A)\Omega_\omega^A$ and that $H^A\Omega_\omega^A=0$. We write

$$\alpha_t^{W_A}(A)=e^{itH^A}Ae^{-itH^A} \tag{4.13}$$

We prove that Ω_ω^A satisfies the α^{W_A}-KMS conditions. In order to show this, we assert that for $z\in D_\beta$ and $A\in\pi_\omega(\mathscr{A}_{A'})$,

$$\alpha_z^{W_A}(A)\Omega_\omega^A=\lim_{n\to\infty}\alpha_z^{W_{A,n}}(A)\Omega_\omega^A \tag{4.14}$$

strongly. To prove this, we expand $\alpha_z^{W_A}(A)$ and $\alpha_z^{W_{A,n}}(A)$ by power series in z, and then we use the method in the proof of Proposition 2.3 and the fact that $\|(\phi_{i+1/2})^n\Omega_\omega^{A,n}\|\leqslant c^m(m!)^{1/2}$ uniformly in n to conclude that (4.14) holds. We leave the detailed proof to the reader. Using (4.14) and the fact that $\Omega_\omega^{A,n}$ satisfies $\alpha^{W_{A,n}}$-KMS conditions, we obtain

$$\begin{aligned}(\Omega_\omega^A, A\alpha_{i\beta}^{W_A}(B)\Omega_\omega^A)&=\lim_{n\to\infty}(\Omega_\omega^{A,n}, A\alpha_{i\beta}^{W_{A,n}}(B)\Omega_\omega^{A,n})\\&=\lim_{n\to\infty}(\Omega_\omega^{A,n}, BA\Omega_\omega^{A,n})=(\Omega_\omega^A, BA\Omega_\omega^A)\end{aligned}$$

for any $A, B\in\bigcup\pi_\omega(\mathscr{A}_{A'})$. Since $\bigcup\pi_\omega(\mathscr{A}_{A'})$ is dense in $\pi_\omega(\mathscr{A})''$, Ω_ω^A satisfies α^{W_A}-KMS conditions. Thus, Ω_ω^A is separating for $\pi(\mathscr{A})''$. Since $\Omega_\omega^{A,n}\in V_\omega$, $\Omega_\omega^A\in V_\omega$ by (4.14). From Proposition 2.5.30 of Ref. 2, it follows that Ω_ω^A is cyclic for $\pi(\mathscr{A})''$. This proves the lemma completely.

Remark. As stated in the remark below Theorem 1.4, it can be proven that any Gibs state $\omega\in\Gamma$ is an even Gibbs state ($\omega\in\Gamma^e$). The proof can be produced in the following manner: Let $\omega\in\Gamma$ and let $A^-\in\mathscr{A}_A^0$ be an odd element. By the Gibbs condition (G-4), $(\Omega_\omega^A, A^-\Omega_\omega^A)=0$. Note that the surface energy W_A is affiliated with $\mathscr{A}_A^e$. Let

$$\Gamma_{i\beta/2}^A(-W_A)\Omega_\omega^A\equiv\exp[-\tfrac{1}{2}\beta(H^A+W_A)]\Omega_\omega^A$$

be the local perturbation of Ω_ω^Λ by $-W_\Lambda$. Then $\Gamma_{i\beta/2}^\Lambda(-W_\Lambda)\Omega_\omega^\Lambda=\Omega_\omega$. Since

$$(\Gamma_{i\beta/2}^\Lambda(-W_\Lambda)\Omega_\omega^\Lambda,\ A^-\Gamma_{i\beta/2}^\Lambda(-W_\Lambda)\Omega_\omega)=0$$

we have that $(\Omega_\omega, A^-\Omega_\omega)=0$. We do not produce the detailed proof here and leave it as an exercise. However, we will not use the above fact in the rest of the paper.

As before, we write that for $\omega\in\Gamma$ and $A\in\pi_\omega(\mathscr{A})''$,

$$\begin{aligned}\Omega_\omega^A&=\Gamma_{i\beta/2}^A\Omega_\omega\\ \alpha_z(A)&=e^{izh}Ae^{-izh}\\ \alpha_z^{W_A}&=e^{iz(h-w_A)}Ae^{-iz(h-W_A)}\end{aligned}\tag{4.15}$$

We introduce the notion of the algebra at infinity. Let Γ^e be the set of even Gibbs states. Let $\mathscr{A}_\Lambda^e$ and $\mathscr{A}^e$ be the algebras of even elements defined in (1.18). For $\omega\in\Gamma^e$, the algebra $\mathscr{B}_\omega^e$ at infinity and the center $\mathscr{Z}_\omega^e$ for $(\mathscr{A}^e,\omega)$ are defined by

$$\mathscr{B}_\omega^e=\bigcap_{\Lambda\text{ finite}}\pi_\omega(\mathscr{A}_{\Lambda^c}^e)'',\qquad \mathscr{Z}_\omega^e=\pi_\omega(\mathscr{A}^e)''\cap\pi_\omega(\mathscr{A}^e)'\tag{4.16}$$

where

$$\mathscr{A}_{\Lambda^c}^e=\Bigg(\bigcup_{\substack{\Lambda'\cap\Lambda=\varnothing\\ \Lambda'\text{ finite}}}\mathscr{A}_{\Lambda'}^e\Bigg)''$$

Then $\mathscr{B}_\omega^e\subset\mathscr{Z}_\omega^e$. For any $\omega\in\Gamma$, the algebra $\mathscr{B}_\omega$ at infinity and the center $\mathscr{Z}_\omega$ for $(\mathscr{A},\omega)$ are defined analogously.

Lemma 4.4. For $\omega\in\Gamma^e$, let $T\in\mathscr{Z}_\omega^e$. Assume that for any finite $\Lambda\subset Z$, $A\in\pi_\omega(\mathscr{A}_\Lambda)$, and $B\in(\mathscr{A}_{\Lambda^c})$

$$(\Omega^\Lambda, ABT\Omega^\Lambda)=\rho_\Lambda(A)(\Omega^\Lambda, BT\Omega^\Lambda)$$

Then $T\in\mathscr{B}_\omega^e$.

Proof. In this proof we suppress ω in the notation. For a given $\omega\in\Gamma^e$, we write that for $\sigma=e$ or 0

$$\mathscr{H}_\sigma=(\pi(\mathscr{A}^\sigma)\Omega^\Lambda)^-,\qquad \mathscr{H}_\sigma^\Lambda=(\pi(\mathscr{A}_\Lambda^\sigma)\Omega^\Lambda)^-,\qquad \mathscr{H}_\sigma^{\Lambda^c}=(\pi(\mathscr{A}_{\Lambda^c}^\sigma)\Omega^\Lambda)^-\tag{4.17}$$

We note that the set

$$\begin{aligned}&\pi(\mathscr{A}_\Lambda)\times\pi\Bigg(\bigcup_{\Lambda'\cap\Lambda=\varnothing}\mathscr{A}_{\Lambda'}\Bigg)\\ &\quad\equiv\Bigg\{\pi(A)\in\pi(\mathscr{A}):\quad A=\sum^n B_iC_i,\quad B_i\in\mathscr{A}_\Lambda,\quad C_i\in\bigcup_{\Lambda'\cap\Lambda=\varnothing}\mathscr{A}_{\Lambda'}\Bigg\}\end{aligned}$$

is dense in $\pi(\mathscr{A})''$. And so one may decompose $\mathscr{H}_e$ by

$$\mathscr{H}_e = \mathscr{H}_{(e,e)} \oplus \mathscr{H}_{(0,0)} \tag{4.18}$$

where $\mathscr{H}_{(e,e)} = \mathscr{H}_e^{\Lambda} \otimes \mathscr{H}_e^{\Lambda^c}$, and

$$\mathscr{H}_{(0,0)} = \left\{ \sum^n A_i B_i \Omega^{\Lambda}: \quad A_i \in \pi(\mathscr{A}_{\Lambda}^0), \quad B_i \in \pi(\mathscr{A}_{\Lambda^c}^0) \right\}^-$$

Let P_1 and P_2 be the orthogonal projections to $\mathscr{H}_{(e,e)}$ and $\mathscr{H}_{(0,0)}$, respectively. Then for A: $\mathscr{H}_e \to \mathscr{H}_e$, we have

$$A = P_1 A P_1 + P_2 A P_2 + P_1 A P_2 + P_2 A P_1$$

Let $T \in \mathscr{Z}_{\omega}^e$ satisfy the assumption in the lemma. Write

$$T = P_1 T P_1 + P_2 T P_2 + P_1 T P_2 + P_2 T P_1$$

For any A_1^+, $A_2^+ \in \pi(\mathscr{A}_{\Lambda}^e)$, B_1^-, $B_2^- \in \pi(\mathscr{A}_{\Lambda}^0)$, C_1^+, $C_2^+ \in \pi(\mathscr{A}_{\Lambda^c}^e)$, and D_1^-, $D_2^- \in \pi(\mathscr{A}_{\Lambda^c}^0)$, we have that by the assumption in the lemma

$$\begin{aligned} ((A_1^+ C_1^+)^* \Omega^{\Lambda}, P_1 T P_2 B_1^- D_1^- \Omega^{\Lambda}) &= (\Omega^{\Lambda}, A_1^+ C_1^+ T B_1^- D_1^- \Omega^{\Lambda}) \\ &= (\Omega^{\Lambda}, A_1^+ B_1^- T C_1^+ D_1^- \Omega^{\Lambda}) \\ &= \rho_{\Lambda}(A_1^+ B_1^-)(\Omega^{\Lambda}, T C_1^+ D_1^- \Omega^{\Lambda}) \\ &= 0 \end{aligned} \tag{4.19}$$

Here we have used the fact that $\rho_{\Lambda}(A^-) = 0$ for $A^- \in \pi(\mathscr{A}_{\Lambda}^0)$. Equation (4.19) implies

$$P_1 T P_2 = 0 \tag{4.20}$$

By a method similar in the above, we also have

$$P_2 T P_1 = 0 \tag{4.21}$$

From the assumption it follows that

$$\begin{aligned} ((A_1^+ C_1^+)^* \Omega^{\Lambda}, P_1 T P_1 A_2^+ C_2^+ \Omega^{\Lambda}) &= (\Omega^{\Lambda}, A_1^+ A_2^+ T C_1^+ C_2^+ \Omega^{\Lambda}) \\ &= \rho_{\Lambda}(A_1^+ A_2^+)(\Omega^{\Lambda}, C_1^+ T C_2^+ \Omega^{\Lambda}) \end{aligned} \tag{4.22}$$

We write $\Omega^{\Lambda} = \Omega_1^{\Lambda} \otimes \Omega_2^{\Lambda}$, where Ω_1^{Λ} and Ω_2^{Λ} are the cyclic vectors for $\pi(\mathscr{A}_{\Lambda}^e)$ and $\pi(\mathscr{A}_{\Lambda^c}^e)$, respectively. Then, from (4.22) we have

$$\begin{aligned} &(A_1^+ \Omega_1^{\Lambda}, (C_1^+ \Omega_2^{\Lambda}, T C_2^+ \Omega_2^{\Lambda}) A_2^+ \Omega_1^{\Lambda}) \\ &\quad = (A_1^+ \Omega_1^{\Lambda}, A_2^+ \Omega_1^{\Lambda})(C_1^+ \Omega_1^{\Lambda} \otimes \Omega_2^{\Lambda}, T C_2^+ \Omega_1^{\Lambda} \otimes \Omega_2^{\Lambda}) \|\Omega_1^{\Lambda}\|^{-2} \end{aligned}$$

The above implies that for any $\phi, \psi \in \mathscr{H}_e^{A^c}$

$$(\phi, T\psi) = b(\phi, \psi) \mathbb{1}_{\mathscr{H}_e^A} \tag{4.23}$$

for some binlinear form $b(\phi, \psi)$. Using the method in the above, we conclude that for any $\tilde{\phi}, \tilde{\psi} \in \mathscr{H}_0^{A^c}$

$$(\tilde{\phi}, T\tilde{\psi}) = \tilde{b}(\tilde{\phi}, \tilde{\psi}) \mathbb{1}_{\mathscr{H}_0^A} \tag{4.24}$$

The relations (4.20), (4.21), (4.23), and (4.24) imply $T \in \pi(\mathscr{A}_{A^c}^e)''$ for all finite $A \subset Z$. This proves that $T \in \mathscr{B}_\omega^e$.

The next step is to show that Γ is a simplex.

Proposition 4.5. The set Γ^e of Gibbs states is a Choquet simplex.

Proof. Let $C = \{t\omega : 0 \leqslant t < \infty, \omega \in \Gamma^e\}$ be the cone through Γ^e. We must show that C is a lattice. Let $\omega_1, \omega_2 \in C$ and define $\omega = \omega_1 + \omega_2$. Let $\hat{\omega}$ be the normal extension of ω to $\pi_\omega(\mathscr{A})''$. Notice that any $\omega' \in \Gamma$ satisfies KMS conditions. Following the proof of Theorem 5.3.30 of Ref. 1, one can show that ω_1 and ω_2 are π_ω-normal, and that there exist positive $T_1, T_2 \in \mathscr{Z}_\omega^e$ such that

$$\omega_1(A) = \hat{\omega}(AT_1), \qquad \omega_2(A) = \hat{\omega}(AT_2)$$

Since ω_1 and ω_2 satisfy condition (G-4), one has that for any $A \in \pi(\mathscr{A}_A)$, $B \in \pi(\mathscr{A}_{A^c})$

$$\hat{\omega}(ABT_i) = \omega_i(AB) = \rho_A(A)\,\omega_i(B) = \rho_A(A)\,\omega(BT_i)$$

Thus, by Lemma 4.4, $T_1, T_2 \in \mathscr{B}_\omega^e$.

Since $\mathscr{B}_\omega^e$ is Abelian, the greatest lower bound $T_1 \wedge T_2 \in \mathscr{B}_\omega^e$ exists. Define

$$(\omega_1 \wedge \omega_2)(A) = \hat{\omega}(A(T_1 \wedge T_2))$$

Then $\omega_1 \wedge \omega_2$ is a unique greatest lower bound for ω_1 and ω_2. Since $T_1 \wedge T_2 \in \mathscr{B}_\omega^e$, $\omega_1 \wedge \omega_2$ satisfies the conditions (G-1)–(G-4), and so $\omega_1 \wedge \omega_2 \in C$.

Proposition 4.6. Let $\{A\}$ be a net tending to Z and let $\omega \in \Gamma$. Then There exist a subnet $\{A'\}$ of $\{A\}$ and a state $\tilde{\omega}$ over $\pi(\mathscr{A})''$ such that $\tilde{\omega}$ is an α-KMS state and for $z \in \overline{D^{(1)}}$ and $A, B \in \bigcup \pi_\omega(\mathscr{A}_A)$

$$\tilde{\omega}(A\alpha_z(B)) = \lim_{A' \to Z} (\Omega_\omega^{A'}, A\alpha_z^{W_{A'}}(B)\Omega_\omega^{A'})/\|\Omega_\omega^{A'}\|^2$$

where α_t is the modular automorphism on $\pi(\mathscr{A})''$ with respect to Ω_ω.

Proof. We write that for any $A, B \in \pi_\omega(\mathscr{A}_{\Lambda''})$ and $z \in \overline{D_\beta^{(1)}}$

$$F_\Lambda(A, B; z) = (\Omega_\omega^\Lambda, A\alpha_z^{W_\Lambda}(B)\Omega_\omega^\Lambda) \tag{4.25}$$

Then $F_\Lambda(A, B; z)$ is analytic on $D_\beta^{(1)}$ and bounded on $\overline{D_\beta^{(1)}}$ by Lemma 4.3 and its proof. Notice that

$$\frac{d^m}{dz^m} F_\Lambda(A, B; z) = i^m(\Omega_\omega^\Lambda, A\alpha_Z^{W_\Lambda}(\delta_{H^\Lambda}^m(B))\Omega_\omega^\Lambda)$$

where H^Λ has been defined in (4.11). Using the method in the proof of Proposition 2.3 and the α^{W_Λ}-KMS conditions for Ω_ω^Λ, one can show that

$$\left|\frac{d^m}{dz^m} F_\Lambda(A, B; z)\right| \leqslant C^m \left(\Omega_\omega^\Lambda, \prod_{i=1}^m |(t + g\phi_{l_i+1/2})| \, \Omega_\omega^\Lambda\right)$$

We next employ the method used in the proof of Proposition 3.5 and the regularity of $\omega \in \Gamma$ to obtain

$$\left(\Omega_\omega^\Lambda, \prod_{i=1}^m |(t + g\phi_{l_i+1/2})| \, \Omega_\omega^\Lambda\right) \leqslant C^m (m!)^{1/2}$$

uniformly in Λ. Thus, one has that

$$\left|\frac{d^m}{dzm} F_\Lambda(A, B; z)\right| \leqslant C^m (m!)^{1/2} \qquad \text{uniformly in } z \text{ and } \Lambda \tag{4.26}$$

for $z \in D_\beta^{(1)}$. Next we note that by the Peierls–Bogoliubov inequality

$$\begin{aligned} \|\Omega_\omega^\Lambda\|^2 &\geqslant \inf_n \|\Omega_\omega^{\Lambda,n}\|^2 \\ &\geqslant \inf_n \exp[-(\Omega_\omega, W_{\Lambda,n}\Omega_\omega)] \\ &= \exp[-(\Omega_\omega, W_\Lambda \Omega_\omega)] \\ &\geqslant e^{-c} \end{aligned} \tag{4.27}$$

uniformly in Λ.

We now choose a subnet $\{\Lambda'\}$ of $\{\Lambda\}$ such that

$$F(A, B; z) = \lim_{\Lambda' \to Z} F_{\Lambda'}(A, B; z) \tag{4.28}$$

Then $F(A, B; z)$ is analytic on $D_\beta^{(1)}$, bounded on $\overline{D_\beta^{(1)}}$, and

$$\frac{d^m}{dz^m} F(A, B; z) = \lim_{\Lambda' \to Z} \frac{d^m}{dz} F_{\Lambda'}(A, B; z)$$
$$\left| \frac{d^m}{dz^m} F(A, B; z) \right| \leqslant C^m (m!)^{1/2} \tag{4.29}$$

From (4.27) and (4.30) it follows that

$$\tilde{\omega}(A) = F(A, \mathbb{1}; 0)/F(\mathbb{1}, \mathbb{1}; 0) \tag{4.30}$$

defines a state over $\pi(\mathscr{A})''$. Let

$$\omega^\Lambda(A) = (\Omega_\omega^\Lambda, A\Omega_\omega^\Lambda)\, \|\Omega_\omega^\Lambda\|^{-2} \tag{4.31}$$

Then from (4.26)–(4.30) one shows that

$$\begin{aligned} \tilde{\omega}(A\delta_h^m(B)) &= \lim_{\Lambda' \to Z} \omega^{\Lambda'}(A\delta_{H^{\Lambda'}}^m(B)) \\ &= \lim_{\Lambda' \to Z} \omega^{\Lambda'}(A\delta_h^m(B)) \end{aligned}$$

Here we have used the fact that for any $B \in \pi_\omega(\mathscr{A}_{\Lambda''})$, $\delta_{H^{\Lambda'}}^m(B) = \delta_h^m(B)$ for sufficiently large Λ. Thus

$$\tilde{\omega}(A\alpha_z(B)) = \lim_{\Lambda' \to Z} \omega^{\Lambda'}(A\alpha_z^{W_{\Lambda'}}(B)) \tag{4.32}$$

for any $A, B \in \bigcup \pi_\omega(\mathscr{A}_\Lambda)$ and $z \in \bar{D}_\beta$. The proposition follows from (4.32) and from the fact that $\omega^{\Lambda'}$ is an $\alpha^{W_{\Lambda'}}$-KMS state.

Lemma 4.7. Let $\omega \in \Gamma$ be an extremal state. Then $\mathscr{B}_\omega = \{c\mathbb{1}\}$.

Proof. First we note that $\mathscr{B}_\omega \subset \mathscr{Z}_\omega$. If there is $h \in \mathscr{B}_\omega$ nonconstant with $0 \leqslant h \leqslant 1$, the state $\omega'(A) = \omega(h)^{-1}\, \omega(hA)$ will be a Gibbs state: Since $h \in \mathscr{B}_\omega$, it follows that for any finite Λ, $A \in \pi_\omega(\mathscr{A}_\Lambda)$, $B \in \pi_\omega(\mathscr{A}_{\Lambda'})$ with $\Lambda' \subset \Lambda^c$

$$\omega^\Lambda(hAB) = \omega^\Lambda(A)\, \omega^\Lambda(hB) = \rho_\Lambda(A)\, \omega^\Lambda(hB)$$

and so ω' satisfies condition (G-4). Obviously ω' satisfies other conditions in Definition 1.2. Similarly the state $\omega''(A) = \omega(1-h)^{-1}\, \omega((1-h)\, A)$ is a Gibbs state. Thus, ω is a convex combination of ω' and ω''.

Finally we prove the uniqueness of Gibs states:

Proof of Theorem 1.4. Let $\omega \in \Gamma^e$ be an extremal Gibbs state and let ρ be a weak*-limit of a sequence $\{\rho_{\Lambda_\alpha}\}$ of finite-volume Gibbs states. By

Proposition 4.6 there exists a subnet $\{\Lambda'_\alpha\}$ of $\{\Lambda_\alpha\}$ such that for any $A \in \pi_\omega(\bigcup \mathscr{A}_\Lambda)$

$$\tilde{\omega}(A) = \lim_{\Lambda'_\alpha \to Z} \omega^{\Lambda'_\alpha}(A) = \rho(A) \tag{4.33}$$

Here we have used the fact that $\omega \in \Gamma$ to obtain the second equality. Since $\tilde{\omega}$ is an α-KMS state, we use Proposition 5.3.29 of Ref. 2 (and its proof) for $\pi(\mathscr{A}^e)$ to conclude that there exists a unique positive operator T affiliated with $\mathscr{Z}^e_\omega$ such that for any $A \in \pi(\mathscr{A}^e)''$

$$\tilde{\omega}(A) = \omega(TA) = (T^{1/2}\Omega_\omega, AT^{1/2}\Omega_\omega) \tag{4.34}$$

The above relation can be extended to $\pi(\mathscr{A})''$. We write that

$$\tilde{\omega}^\Lambda(A) = (T^{1/2}\Gamma^\Lambda_{i\beta/2}\Omega_\omega, AT^{1/2}\Gamma^\Lambda_{i\beta/2}\Omega_\omega) \tag{4.35}$$

From condition (G-4) and (4.33) it follows that

$$\tilde{\omega}(A\alpha_z(B)) = (\Omega_\rho, \pi_\rho(A)\, \alpha_z(\pi_\rho(B))\Omega_\rho)$$

for any $A \in \mathscr{A}$, $B \in \bigcup \mathscr{A}_\Lambda$, $z \in D_\beta$, and so

$$\tilde{\omega}^\Lambda(A) = (\Gamma^\Lambda_{i\beta/2}\Omega_\rho, A\Gamma^\Lambda_{i\beta/2}\Omega_\rho) \equiv \rho^\Lambda(A) \tag{4.36}$$

Thus, using (4.36) and the fact that ω, $\rho \in \Gamma$, we obtain

$$\begin{aligned}\omega^\Lambda(ABT) = \tilde{\omega}^\Lambda(AB) = \rho^\Lambda(AB) = \rho_\Lambda(A)\, \rho^\Lambda(B) &= \rho_\Lambda(A)\, \tilde{\omega}^\Lambda(B) \\ &= \rho_\Lambda(A)\, \omega^\Lambda(BT)\end{aligned} \tag{4.37}$$

By Lemma 4.4 and (4.37), T is affiliated with $\mathscr{B}_\omega$.

Since $\mathscr{B}_\omega = C\mathbb{1}$ by Lemma 4.7, $\omega = \rho$. That is, if ρ is a weak*-limit of finite-volume Gibbs states ρ_Λ and if ω and ω' are extremal states in Γ, then $\omega = \rho = \omega'$. This proves Theorem 1.4 completely.

Remark. The proof can be shortened by the following argument: Let T be the positive operator satisfying (4.34). Since T and the modular operator Δ commute strongly by Proposition 5.3.28 of Ref. 2, $T^{1/2}\Omega_\omega = \Delta^{1/2}T^{1/2}\Omega_\omega \in V_\omega$, where V_ω is the natural cone. Since $T^{1/2}\Omega_\omega$ is separating for $\pi_\omega(\mathscr{A})''$, $T^{1/2}\Omega_\omega$ is cyclic for $\pi(\mathscr{A})''$ by Proposition 2.5.30 of Ref. 2. Thus, $(\mathscr{H}_\omega, \pi_\omega, T^{1/2}\Omega_\omega)$ is the cyclic representation with respect to $\tilde{\omega}$, and so $\tilde{\omega}^\Lambda(A) = \rho_\Lambda(A)\, \tilde{\omega}(\mathbb{1})$ by the condition (G-4). Hence (4.37) holds.

Proof of Theorem 1.5. The follows from Theorem 1.4, Lemma 4.7, and Theorem 2.6.10 of Ref. 2.

5. DISCUSSION

So far we have proved the uniqueness and the cluster property of Gibbs states for the semiclassical approximation (1.3) of the full quantum polyacetylene model (1.1) for any $\beta > 0$ with free boundary conditions for boson fields. Thus, within the above simplification, it is impossible to construct a soliton sector, and so the heuristic arguments in Refs. 10 and 11 would fail. Therefore, it would be interesting to know whether our results in this paper can be extended to the following cases:

(a) Other boundary conditions.
(b) Full quantum model (1.1).
(c) Ground states ($\beta = \infty$).

In the rest of this section we give a brief discussion of the above cases.

We first consider cases (a) and (b). For $\Lambda = \{n,..., m\} \subset Z$ and for any $f: \Lambda \to \mathbb{R}$, define

$$P(f) = \sum_{i \in \Lambda} P_i f(i), \qquad U(\partial^* f) = \sum_{i=1}^{m-1} (u_{i+1} - u_i) f(i) \tag{5.1}$$

and introduce two norms

$$\begin{aligned} \|f\|_1^{\sim} &= \sum_{\substack{i,j \in \Lambda \\ i \neq j}} [1 + (i-j)^2]^{-1} |f(j)| \leqslant \text{const} \times \|f\|_1 \\ \|f\|_2^{\sim} &= \sum_{\substack{i,j \in \Lambda \\ i \neq j}} [1 + (i-j)^2]^{-1} |f(i) f(j)| \end{aligned} \tag{5.2}$$

Then for the full quantum model with $\beta > 0$ it can be shown that

$$\rho_\Lambda(\exp[p(h_1) + u(\partial^* h_2)]) \leqslant \exp\left\{c \sum_{i=1}^{2} [\|h_i\|_1^{\sim} + (\|h_i\|_2^{\sim})^2]\right\} \tag{5.3}$$

for some constant c independent of Λ, where ρ_Λ is the finite Gibbs state for the quantum model (1.1) with some boundary conditions (e.g., Dirichlet or free boundary conditions). A bound similar to that in (5.3) also holds for the semiclassical approximation with other boundary conditions. Because of a lack of space, we will not produce the proof of the bound (5.3) here.

In case (b) the bound (5.3) implies that any infinite-volume-limit equilibrium state is entire analytic and a modular state on a quasilocal algebra. And so a quantum dynamical system can be constructed. But in order to show the uniqueness of equilibrium states, one has to formulate an appropriate Gibbs condition corresponding to Definition 1.2 (G-4). If one removes the surface energy W_Λ defined in (1.17) $[\phi_i = (\partial u)_i]$, the perturbed states

$$\omega^\Lambda(A) = (\Gamma^\Lambda_{i\beta/2}\Omega_\omega, A\Gamma^\Lambda_{i\beta/2}\Omega_\omega)$$

are not factorized. Thus, for $\Lambda = \{n,..., m\}$ one has to include either

$$W'_\Lambda = [(u_n - u_{n-1})^2 + (u_{m+1} - u_m)^2]\, w^2/2$$

or else

$$w''_\Lambda = (2u_n u_{n-1} + 2u_{m+1} u_m)\, w^2/2$$

into the surface energy to factorize ω^Λ. Apparently W'_Λ and W''_Λ are too singular to define $\Gamma^\Lambda_{i\beta/2}\Omega_\omega$ in terms of the Dyson expansion (3.7). A difficulty similar to the above arises in case (a). Because of this difficulty we are unable to extend our results to cases (a) and (b). On the other hand, the results in Ref. 5 suggest that the uniqueness of equilibrium states may hold for (a) and (b).

Finally, we consider case (c). For $\beta = \infty$ (ground states) it can be argued that the quantum model (1.1) is very closely related to the two-dimensional Yukawa model in $Z \times R$.[5,10,11] Thus, for sufficiently large g, a first-order phase transition may take place. In this case one can construct a soliton sector and prove the existence of fractional charges. We hope to come back to this subject in future.

ACKNOWLEDGMENTS

It is a pleasure to thank Michael Aizenman, Ola Bratteli, and Keiichi Ito for stimulating discussions and helpful suggestions. This work was supported in part by the Korean Science Foundation.

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